

# CARDINAL INVARIANTS DISTINGUISHING PERMUTATION GROUPS

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**ABSTRACT.** We prove that for infinite cardinals  $\kappa < \lambda$  the alternating group  $\text{Alt}(\lambda)$  (of even permutations) of  $\lambda$  is not embeddable into the symmetric group  $\text{Sym}(\kappa)$  (of all permutations) of  $\kappa$ . To prove this fact we introduce and study several monotone cardinal group invariants which take value  $\kappa$  on the groups  $\text{Alt}(\kappa)$  and  $\text{Sym}(\kappa)$ .

By Cayley's classical theorem [8, 1.6.8], each group  $G$  embeds into the group  $\text{Sym}(|G|)$  of all bijective transformations of the cardinal  $|G|$ . Observe that for a symmetric group  $G = \text{Sym}(\kappa)$  on an infinite cardinal  $\kappa$  Cayley's Theorem can be improved: the group  $G = \text{Sym}(\kappa)$  embeds into the symmetric group  $\text{Sym}(\log |G|)$ . This suggests the following question: *can each infinite group  $G$  be embedded into the symmetric group  $\text{Sym}(\log |G|)$ ?* Here for a cardinal  $\kappa$  by  $\log(\kappa) = \min\{\lambda : \kappa \leq 2^\lambda\}$  we denote the *logarithm* of  $\kappa$ . Another question of the same flavor asks: *can the symmetric group  $\text{Sym}(\kappa)$  on an infinite cardinal  $\kappa$  be embedded into the symmetric group  $\text{Sym}(\lambda)$  on a smaller cardinal  $\lambda < \kappa$ ?* In this paper we shall give negative answers to both questions. First, we need to introduce some notation.

Let  $\kappa$  be a (finite or infinite) cardinal. By  $\text{Sym}(\kappa)$  we denote the set of bijective functions from  $\kappa$  to  $\kappa$ , also called the *permutations* of  $\kappa$ . The set  $\text{Sym}(\kappa)$  endowed with the operation of composition of permutations is a group called the *symmetric group* on  $\kappa$ . This group contains a normal subgroup  $\text{Sym}_{\text{fin}}(\kappa)$  consisting of permutations  $f : \kappa \rightarrow \kappa$  with finite support  $\text{supp}(f) = \{x \in \kappa : f(x) \neq x\}$ . A permutation  $f : \kappa \rightarrow \kappa$  with two-element support  $\text{supp}(f)$  is called a *transposition* of  $\kappa$ . It is well-known that each finitely supported permutation can be written as a finite composition of transpositions. A permutation  $f : \kappa \rightarrow \kappa$  is called *even* if it can be written as the composition of an even number of transpositions. The even permutations form a subgroup  $\text{Alt}(\kappa)$  of  $\text{Sym}(\kappa)$  called the *alternating group* on  $\kappa$ . It is a normal subgroup of index 2 in  $\text{Sym}_{\text{fin}}(\kappa)$ . So, we get the inclusions

$$\text{Alt}(\kappa) \subset \text{Sym}_{\text{fin}}(\kappa) \subset \text{Sym}(\kappa).$$

For an infinite cardinal  $\kappa$  these groups have cardinalities  $|\text{Alt}(\kappa)| = |\text{Sym}_{\text{fin}}(\kappa)| = \kappa$  and  $|\text{Sym}(\kappa)| = 2^\kappa$ .

The following theorem answers in negative the questions posed at the beginning of the paper.

**Theorem 1.** *Let  $\kappa < \lambda$  be two infinite cardinals. Then there is no embedding of  $\text{Alt}(\lambda)$  into  $\text{Sym}(\kappa)$ .*

The idea of the proof of this theorem is rather natural: find a cardinal characteristic  $\varphi(G)$  of a group  $G$  which is invariant under isomorphisms of groups, is monotone under taking subgroups, and takes value  $\kappa$  on the groups  $\text{Alt}(\kappa)$  and  $\text{Sym}(\kappa)$ . Then for any infinite cardinals  $\kappa < \lambda$  we would have  $\varphi(\text{Alt}(\lambda)) = \lambda \not\leq \kappa = \varphi(\text{Sym}(\kappa))$ , and the monotonicity of  $\varphi$  would imply that  $\text{Alt}(\lambda)$  does not embed into  $\text{Sym}(\kappa)$ . In the sequel, cardinal characteristics of groups which are invariant under isomorphisms of groups will be called *cardinal group invariants*. A cardinal group invariant  $\varphi$  is called *monotone* if  $\varphi(H) \leq \varphi(G)$  for any subgroup  $H$  of a group  $G$ .

Many examples of monotone cardinal group invariants can be produced as minimizations of cardinal characteristics of (semi)topological groups over certain families of admissible topologies on a given group. Now we explain this approach in more details. First, we define four families  $\mathcal{T}_s(G)$ ,  $\mathcal{T}_c(G)$ ,  $\mathcal{T}_g(G)$ , and  $\mathcal{T}_l(G)$  of admissible topologies on a group  $G$ .

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A topology  $\tau$  on a group  $G$  is called *shift-invariant* if for every  $a, b \in G$  the two-sided shift  $s_{a,b} : G \rightarrow G$ ,  $s_{a,b} : x \mapsto axb$ , is a homeomorphism of the topological space  $(G, \tau)$ . This is equivalent to saying that the group multiplication  $G \times G \rightarrow G$ ,  $(x, y) \mapsto xy$ , is separately continuous. A group endowed with a shift-invariant topology is called a *semitopological group*, (see [1]). For a group  $G$  by  $\mathcal{T}_s(G)$  we denote the family of all Hausdorff shift-invariant topologies on  $G$ .

We shall say that a topology  $\tau$  on a group  $G$  has *separately continuous commutator* if the function  $G \times G \rightarrow G$ ,  $(x, y) \mapsto xyx^{-1}y^{-1}$ , is separately continuous. By  $\mathcal{T}_c(G)$  we denote the family of Hausdorff shift-invariant topologies on  $G$  having separately continuous commutator.

A topology  $\tau$  on a group  $G$  is called a *group topology* if the function  $G \times G \rightarrow G$ ,  $(x, y) \mapsto xy^{-1}$ , is jointly continuous. By  $\mathcal{T}_g(G)$  we denote the family of Hausdorff group topologies on  $G$ .

A group topology  $\tau$  on a group  $G$  is called *linear* if it has a neighborhood base at the unit  $1_G$  of  $G$  consisting of  $\tau$ -open subgroups of  $G$ . By  $\mathcal{T}_l(G)$  we denote the family of linear Hausdorff group topologies on  $G$ .

It follows that  $\mathcal{T}_s(G) \supset \mathcal{T}_c(G) \supset \mathcal{T}_g(G) \supset \mathcal{T}_l(G)$  for every group  $G$ .

By a *cardinal topological invariant of semitopological groups* we understand a function  $\varphi$  assigning to each semitopological group  $G$  some cardinal  $\varphi(G)$  so that  $\varphi(G) = \varphi(H)$  for any topologically isomorphic semitopological groups  $G, H$ . We shall say that the function  $\varphi$  is *monotone* if  $\varphi(H) \leq \varphi(G)$  for any subgroup  $H$  of a semitopological group  $G$ .

Any (monotone) cardinal topological invariant  $\varphi$  of semitopological groups induces four (monotone) cardinal group invariants  $\varphi_s, \varphi_c, \varphi_g, \varphi_l$  assigning to each group  $G$  the cardinals

- $\varphi_s(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_s(G)\}$ ,
- $\varphi_c(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_c(G)\}$ ,
- $\varphi_g(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_g(G)\}$ ,
- $\varphi_l(G) = \min\{\varphi(G, \tau) : \tau \in \mathcal{T}_l(G)\}$ .

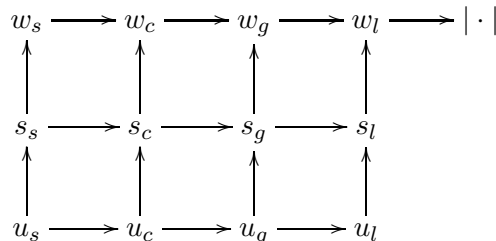
The inclusions  $\mathcal{T}_s(G) \supset \mathcal{T}_c(G) \supset \mathcal{T}_g(G) \supset \mathcal{T}_l(G)$  imply the inequalities

$$\varphi_s(G) \leq \varphi_c(G) \leq \varphi_g(G) \leq \varphi_l(G).$$

We shall apply this construction to three cardinal topological invariants of semitopological groups: the weight  $w$ , the spread  $s$  and the uniform spread  $u$ . A subset  $D$  of a semi-topological group  $(G, \tau)$  is called *uniformly discrete* if there exists a  $\tau$ -open neighborhood  $U \subset G$  of the unit  $1_G$  such that  $y \notin xU$  for any distinct points  $x, y \in G$ . For a semi-topological group  $(G, \tau)$  let

- $w(G, \tau) = \min\{|\mathcal{B}| : \mathcal{B} \subset \tau \text{ is a base of the topology } \tau\}$  be the *weight* of  $(G, \tau)$ ;
- $s(G, \tau) = \sup\{|D| : D \subset G \text{ is a discrete subspace of } (G, \tau)\}$  be the *spread* of  $(G, \tau)$  and
- $u(G, \tau) = \sup\{|D| : D \subset G \text{ is a uniformly discrete subset of } (G, \tau)\}$  be the *uniform spread* of  $(G, \tau)$ .

Observe that the weight and spread of  $(G, \tau)$  depend only on the topology  $\tau$  whereas the definition of the uniform spread involves both structures (algebraic and topological) of the semitopological group  $(G, \tau)$ . Taking into account that each uniformly discrete subset of a semi-topological group  $(G, \tau)$  is discrete, we conclude that  $u(G) \leq s(G) \leq w(G)$ . Theorem 5.5 of [4] implies that  $w(G) \leq 2^{s(G)}$ . It is easy to see that  $u, s, w$  are monotone cardinal topological invariants of semitopological groups. Minimizing these cardinal functions over the families  $\mathcal{T}_s(G)$ ,  $\mathcal{T}_c(G)$ ,  $\mathcal{T}_g(G)$  and  $\mathcal{T}_l(G)$  we obtain 12 monotone cardinal group invariants that relate as follows. In the diagram an arrow  $\varphi \rightarrow \psi$  between two cardinal group invariants  $\varphi, \psi$  indicates that  $\varphi(G) \leq \psi(G)$  for any group  $G$ .



Next, we define a combinatorial cardinal group invariant  $cn(G)$  called the *weak compatibility number* of a group  $G$ . It is defined as the smallest infinite cardinal  $\kappa$  for which the group  $G$  satisfies the *weak  $\kappa^+$ -compatibility condition*:

- for any finite group  $F$ , and isomorphisms  $f_i : F \rightarrow F_i$ ,  $i < \kappa$ , onto finite subgroups  $F_i$  of  $G$  there are two indices  $i < j < \kappa^+$  and a homomorphism  $\phi : \langle F_i \cup F_j \rangle \rightarrow F$  such that  $\phi \circ f_i = \phi \circ f_j = \text{id}_F$ .

Here by  $\kappa^+$  we denote the successor cardinal of  $\kappa$  and for a subset  $A \subset G$  by  $\langle A \rangle$  we denote the subgroup of  $G$  generated by  $A$ . The weak  $\kappa^+$ -compatibility condition is a weak version of a notion introduced in [7, Def. 1.9]. The definition of the weak compatibility number implies that it is a monotone cardinal group invariant. Observe that each torsion-free group  $G$  has  $cn(G) = \omega$ , so  $cn(G)$  can be much smaller than the cardinal  $s_s(G) \geq \log \log |G|$ .

In Proposition 3 we shall present an algebraic description of the linear weight  $w_l$  and using this description will prove that  $cn(G) \leq w_l(G)$  for any group  $G$ . This inequality allows us to add the weak compatibility number  $cn$  to the diagram describing the relations between cardinal group invariants and obtain the diagram:

$$\begin{array}{ccccccc}
 w_s & \longrightarrow & w_c & \longrightarrow & w_g & \longrightarrow & w_l \longleftarrow cn. \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 s_s & \longrightarrow & s_c & \longrightarrow & s_g & \longrightarrow & s_l \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 u_s & \longrightarrow & u_c & \longrightarrow & u_g & \longrightarrow & u_l
 \end{array}$$

The following theorem implies Theorem 1 and can be considered as a main result of this paper.

**Theorem 2.** *For any infinite cardinal  $\kappa$  and a group  $G$  with  $\text{Alt}(\kappa) \subset G \subset \text{Sym}(\kappa)$  we get*

$$\kappa = s_s(G) = u_c(G) = w_l(G) = cn(G).$$

The proof of this theorem will be divided into three Lemmas 11, 13, 18. We start our proofs with an algebraic description of the linear weight  $w_l(G)$  of a group  $G$ .

**Proposition 3.** *For an infinite group  $G$  its linear weight  $w_l(G)$  is equal to the smallest cardinal  $\kappa$  for which there are subgroups  $G_i$ ,  $i \in \kappa$ , of index  $|G/G_i| \leq \kappa$  such that  $\bigcap_{i \in \kappa} G_i$  coincides with the trivial subgroup  $\{1_G\}$  of  $G$ .*

*Proof.* Let  $w'_l(G)$  denote the smallest cardinal  $\kappa$  such that  $\{1_G\} = \bigcap_{i \in \kappa} G_i$  for some subgroups  $G_i$  of index  $|G/G_i| \leq \kappa$  in  $G$ . We need to prove that  $w_l(G) = w'_l(G)$ .

To prove that  $w'_l(G) \leq w_l(G)$ , use the definition of the linear weight  $w_l(G)$  and find a linear group topology  $\tau$  of weight  $\kappa = w_l(G)$  on  $G$ . Let  $\mathcal{B} \subset \tau$  be a base of the topology  $\tau$  of cardinality  $|\mathcal{B}| = \kappa$ . Let  $\mathcal{B}_1 = \{B \in \mathcal{B} : 1_G \in B\}$  be the neighborhood base at the unit  $1_G$  of the group  $G$ . Since  $|\mathcal{B}_1| \leq |\mathcal{B}| \leq \kappa$ , the set  $\mathcal{B}_1$  can be enumerated as  $\mathcal{B}_1 = \{B_i\}_{i < \kappa}$ . Since the topology is linear, each set  $B_i \in \mathcal{B}_1$  contains an open subgroup  $H_i$  of  $G$ . Taking into account that the family  $\{xH_i : x \in G\}$  is disjoint and each coset  $xH_i$ ,  $x \in G$ , contains some basic set  $U \in \mathcal{B}$ , we conclude that  $|\{xH_i : x \in G\}| \leq |\mathcal{B}| = \kappa$  and hence the subgroup  $H_i$  has index  $\leq \kappa$  in  $G$ . The Hausdorff property of the topology  $\tau$  guarantees that  $\{1_G\} = \bigcap \mathcal{B}_1 = \bigcap_{i < \kappa} H_i$ . So, the family  $\{H_i\}_{i < \kappa}$  witnesses that  $w'_l(G) \leq \kappa = w_l(G)$ .

Now we check that  $w_l(G) \leq w'_l(G)$ . By the definition of the cardinal  $\kappa = w'_l(G)$ , there exists a family  $\mathcal{H}$  of subgroups of index  $\leq \kappa$  in  $G$  such that  $|\mathcal{H}| \leq \kappa$  and  $\{1_G\} = \bigcap \mathcal{H}$ . Observe that for any subgroup  $H \in \mathcal{H}$ , any  $x \in G$ , and any  $y \in xH$ , we get  $xHx^{-1} = yHy^{-1}$ , which implies that the family  $\{xHx^{-1} : x \in G\}$  has cardinality  $\leq |G/H| \leq \kappa$ . Then the family

$$\mathcal{U} = \left\{ \bigcap_{i=1}^n x_i H_i x_i^{-1} : H_1, \dots, H_n \in \mathcal{H}, x_1, \dots, x_n \in G \right\}$$

has cardinality  $|\mathcal{U}| \leq \kappa$  and consists of subgroups of index  $\leq \kappa$  in  $G$ . Now consider the topology  $\tau$  on  $G$  consisting of sets  $U \subset G$  such that for every point  $x \in U$  there is a subgroup  $H \in \mathcal{U}$  such that  $xH \subset U$ .

Using Theorem 1.3.2 in [1], it can be shown that the topology  $\tau$  turns  $G$  into a linear topological group of weight  $\leq \kappa$ . Then  $w_l(G) \leq \kappa = w'_l(G)$ .  $\square$

**Proposition 4.** *Each infinite group  $G$  has  $w_g(G) \leq w_l(G) \leq w_g(G)^\omega$ .*

*Proof.* The inequality  $w_g(G) \leq w_l(G)$  is trivial. To prove that  $w_l(G) \leq w_g(G)^\omega$ , fix a Hausdorff group topology  $\tau$  on  $G$  such that  $w(G, \tau) = w_g(G)$ . Let  $\kappa = w_g(G) = w(G, \tau)$  and fix a neighborhood base  $\{U_\alpha\}_{\alpha \in \kappa} \subset \tau$  at the unit  $1_G$  of the group  $G$ . For every  $\alpha \in \kappa$  put  $U_{\alpha,0} = U_\alpha$  and for every  $n \in \mathbb{N}$  choose a symmetric neighborhood  $U_{\alpha,n} = U_{\alpha,n-1}^{-1} \in \tau$  of the unit  $1_G$  such that  $U_{\alpha,n}U_{\alpha,n} \subset U_{\alpha,n-1}$ . It is easy to see that the intersection  $H_\alpha = \bigcap_{n \in \mathbb{N}} U_{\alpha,n}$  is a subgroup of  $G$ . We claim that this subgroup has index  $|G/H_\alpha| \leq \kappa^\omega$  in  $G$ . Let  $q : G \rightarrow G/H_\alpha := \{xH_\alpha : x \in G\}$ ,  $q : x \mapsto xH_\alpha$ , be the quotient map and  $s : G/H_\alpha \rightarrow G$  be any function such that  $q \circ s = \text{id}$ . For every  $n \in \mathbb{N}$  choose a maximal subset  $K_n \subset G$  such that  $xU_{\alpha,n} \cap yU_{\alpha,n} = \emptyset$  for any distinct points  $x, y \in K_n$ , and observe that  $K_n$  is a discrete subspace of  $(G, \tau)$ , which implies that  $|K_n| \leq w(G, \tau) = w_g(G) = \kappa$ . By the maximality of  $K_n$ , for every  $x \in G$  there is a point  $f_n(x) \in K_n$  such that  $xU_{\alpha,n} \cap f_n(x)U_{\alpha,n} \neq \emptyset$  and hence  $x \in f_n(x)U_{\alpha,n}U_{\alpha,n}^{-1} \subset f_n(x)U_{\alpha,n-1}$ . The functions  $f_n$ ,  $n \in \mathbb{N}$ , form the function  $f = (f_n)_{n=1}^\infty : G \rightarrow \prod_{n=1}^\infty K_n$ . We claim that the composition  $f \circ s : G/H_\alpha \rightarrow \prod_{n \in \mathbb{N}} K_n$  is injective. Given any cosets  $X, Y \in G/H_\alpha$  with  $f \circ s(X) = f \circ s(Y)$ , consider the points  $x = s(X)$  and  $y = s(Y)$ . The equality  $f(x) = f \circ s(X) = f \circ s(Y) = f(y)$  implies that  $f_n(x) = f_n(y)$  for all  $n \in \mathbb{N}$  and hence  $x, y \in f_n(x)U_{\alpha,n-1}$ , which implies that  $x^{-1}y \in U_{\alpha,n-1}^{-1}U_{\alpha,n-1} \subset U_{\alpha,n-2}$  for every  $n \geq 2$ . Then  $x^{-1}y \in \bigcap_{n=2}^\infty U_{\alpha,n-2} = H_\alpha$  and hence  $X = xH_\alpha = yH_\alpha = Y$ , witnessing that the function  $f \circ s : G/V_\alpha \rightarrow \prod_{n \in \mathbb{N}} K_\alpha$  is injective and hence  $|G/V_\alpha| \leq |\prod_{n \in \mathbb{N}} K_\alpha| \leq \kappa^\omega$ . Since  $1_G \in \bigcap_{\alpha \in \kappa} H_\alpha \subset \bigcap_{\alpha \in \kappa} U_\alpha = \{1_G\}$ , we can apply Proposition 3 and conclude that  $w_l(G) \leq \kappa^\omega = w_g(G)^\omega$ .  $\square$

Modifying the proofs of Propositions 3 and 4 we can prove the following two facts about the cardinal group invariant  $u_l$ .

**Proposition 5.** *For an infinite group  $G$  its linear uniform spread  $u_l(G)$  is equal to the smallest cardinal  $\kappa$  such that  $\{1_G\} = \bigcap \mathcal{H}$  for some family  $\mathcal{H}$  of subgroups of index  $\leq \kappa$  in  $G$ .*

**Proposition 6.** *Each infinite group  $G$  has  $u_g(G) \leq u_l(G) \leq u_g(G)^\omega$ .*

Both inequalities in this Propositions 4 and 6 can be strict.

**Example 7.** *The discrete group  $G = \mathbb{Z}$  of integers has  $u_g(G) = u_l(G) = w_g(G) = w_l(G) = \omega < \omega^\omega$ .*

**Example 8.** *Let  $M$  be the unit interval or the unit circle. The homeomorphism group  $G$  of  $M$  has  $u_g(G) = w_g(G) = \omega$  and  $u_l(G) = w_l(G) = \mathfrak{c}$ .*

*Proof.* The inequality  $w_g(G) = \omega$  follows from the fact that the compact-open topology on the homeomorphism group  $G$  is metrizable and separable. The equality  $u_l(G) = \mathfrak{c}$  follows from Proposition 5 and Theorem 6 of [9] saying that the subgroup  $G_+ \subset G$  of orientation preserving homeomorphisms of  $M$  is the only subgroup of index  $< \mathfrak{c}$  in  $G$ .  $\square$

The following proposition gives an upper bound on the weak compatibility number  $cn$ .

**Proposition 9.** *Each infinite group  $G$  has  $cn(G) \leq w_l(G)$ .*

*Proof.* It suffices to prove that  $G$  satisfies the weak  $\kappa^+$ -compatibility condition for  $\kappa = w_l(G)$ . By Proposition 3, there is a family  $\mathcal{H}$  of subgroups of index  $\leq \kappa$  in  $G$  such that  $|\mathcal{H}| \leq \kappa$  and  $\{1_G\} = \bigcap \mathcal{H}$ . Replacing  $\mathcal{H}$  by a larger family, we can assume that for any subgroups  $H_1, \dots, H_n \in \mathcal{H}$  and any points  $x_1, \dots, x_n \in G$  the subgroup  $\bigcap_{i=1}^n x_i H_i x_i^{-1}$  belongs to the family  $\mathcal{H}$ .

To show that the group  $G$  satisfies the weak  $\kappa^+$ -compatibility condition, fix a finite group  $F$  and isomorphisms  $f_i : F \rightarrow F_i$ ,  $i < \kappa^+$ , onto finite subgroups  $F_i \subset G$ . Since the family  $\mathcal{H}$  is closed under finite intersections, for every  $i < \kappa^+$  we can choose a subgroup  $H_i \in \mathcal{H}$  such that  $F_i \cap H_i = \{1_G\}$ . Replacing  $H_i$  by the subgroup  $\bigcap_{x \in F_i} x H_i x^{-1}$ , we can assume that  $x H_i x^{-1} = H_i$  for all points  $x \in F_i$ . Since  $|\mathcal{H}| \leq \kappa < \kappa^+$ , for some subgroup  $H \in \mathcal{H}$  the set  $I = \{i \in \kappa^+ : H_i = H\}$  has cardinality

$|I| = \kappa^+$ . Consider the family of left cosets  $G/H = \{xH : x \in G\}$  and the quotient map  $q : G \rightarrow G/H$ ,  $q : x \mapsto xH$ . Since  $|(G/H)^F| \leq \kappa < \kappa^+$ , there are two indices  $i < j$  in  $I$  such that  $q \circ f_i = q \circ f_j$ . Now consider the subgroup  $F_{ij} = \langle F_i \cup F_j \rangle$  generated by the set  $F_i \cup F_j$ . Taking into account that  $xHx^{-1} = H$  for all  $x \in F_i \cup F_j$ , we conclude that  $xHx^{-1} = H$  for all  $x \in F_{ij}$ , which implies that  $L = F_{ij} \cdot H = \{xy : x \in F_{ij}, y \in H\}$  is a subgroup of  $G$  and  $H$  is a normal subgroup in  $L$ . Consequently, the subspace  $L/H = \{xH : x \in L\}$  has the structure of a group and the restriction  $q|_L : L \rightarrow L/H$  is a group homomorphism. The equality  $q \circ f_i = q \circ f_j$  implies that  $L/H = q(F_{ij}) = q(F_i) = q(F_j)$ . Since  $H \cap F_i = \{1_G\}$ , the restriction  $\varphi = q|_{F_i} : F_i \rightarrow L/H$ , being injective and surjective, is an isomorphism. Then  $\psi = f_i^{-1} \circ \varphi^{-1} : L/H \rightarrow F$  is an isomorphism too. It can be shown that the homomorphism  $\phi = \psi \circ q|_{F_{ij}} : F_{ij} \rightarrow F$  has the required property:  $\phi \circ f_j = \phi \circ f_i = \text{id}_F$ .  $\square$

**Question 10.** *Is  $cn(G) \leq w_g(G)$  for any group  $G$ ?*

Now we are able to prove two equalities of Theorem 2.

**Lemma 11.** *For every infinite cardinal  $\kappa$  and group  $G$  with  $\text{Alt}(\kappa) \subset G \subset \text{Sym}(\kappa)$  we get*

$$\kappa = cn(G) = w_l(G).$$

*Proof.* By Proposition 9,  $cn(G) \leq w_l(G)$ . Since the cardinal group invariants  $cn$  and  $w_l$  are monotone, it suffices to prove that  $w_l(\text{Sym}(\kappa)) \leq \kappa$  and  $cn(\text{Alt}(\kappa)) \geq \kappa$ .

To see that  $w_l(\text{Sym}(\kappa)) \leq \kappa$ , for every  $i \in \kappa$  consider the subgroup  $G_i = \{f \in \text{Sym}(\kappa) : f(i) = i\}$  and observe that the index of this subgroup in  $\text{Sym}(\kappa)$  is equal to  $\kappa$ . Taking into account that  $\bigcap_{i \in \kappa} G_i = \{\text{id}\}$  is the trivial subgroup of  $\text{Sym}(\kappa)$ , and applying Proposition 3, we conclude that  $w_l(\text{Sym}(\kappa)) \leq \kappa$ .

To see that  $cn(\text{Alt}(\kappa)) \geq \kappa$ , it suffices to check that for every  $\lambda < \kappa$  the group  $\text{Alt}(\kappa)$  does not satisfy the weak  $\lambda^+$ -compatibility condition. For every ordinal  $3 \leq i < \lambda^+ \leq \kappa$  consider the 4-element subset  $K_i = \{0, 1, 2, i\}$  and its alternating group  $\text{Alt}(K_i) \subset \text{Alt}(\kappa)$ . Fix an isomorphism  $f_i : \text{Alt}(4) \rightarrow \text{Alt}(K_i)$  and observe that for any  $3 \leq i < j < \lambda^+$  the subgroup  $\langle \text{Alt}(K_i) \cup \text{Alt}(K_j) \rangle$  is equal to  $\text{Alt}(K_i \cup K_j)$  and is isomorphic to the alternating group  $\text{Alt}(5)$ . Since  $\text{Alt}(5)$  is a simple group and  $\text{Alt}(4)$  is not simple there does not exist a surjective homomorphism from  $\langle \text{Alt}(H_i) \cup \text{Alt}(H_j) \rangle$  to  $\text{Alt}(4)$ , which implies that the group  $\text{Alt}(\kappa)$  does not satisfy the weak  $\lambda^+$ -compatibility condition.  $\square$

We say that a topological space  $X$  is  $\sigma$ -discrete if  $X$  can be written as a countable union of discrete subspaces. By [2] or [3, 6.1], for every cardinal  $\kappa$  the group  $\text{Sym}_{\text{fin}}(\kappa)$  is  $\sigma$ -discrete in each shift-invariant Hausdorff topology on  $\text{Sym}_{\text{fin}}(\kappa)$ . The same fact is true for the alternating group  $\text{Alt}(\kappa)$ .

**Theorem 12.** *The alternating group  $\text{Alt}(\kappa)$  on a cardinal  $\kappa$  is  $\sigma$ -discrete in any Hausdorff shift-invariant topology  $\tau$  on  $\text{Alt}(\kappa)$ .*

*Proof.* If the cardinal  $\kappa$  is finite, then the alternating group  $\text{Alt}(\kappa)$  is finite and hence discrete in the topology  $\tau$ . So, we assume that  $\kappa$  is infinite. To prove the proposition, it suffices to check that for every  $n \in \omega$  the subspace  $\text{Alt}_n(\kappa) = \{f \in \text{Alt}(\kappa) : |\text{supp}(f)| = n\}$  of  $\text{Alt}(\kappa)$  is discrete. Given any permutation  $f \in \text{Alt}_n(\kappa)$  we shall construct an open set  $O_f \in \tau$  such that  $f \in O_f \cap \text{Alt}_n(\kappa) \subset \{g \in \text{Alt}(\kappa) : \text{supp}(g) = \text{supp}(f)\}$ .

Choose two disjoint subsets  $A, B \subset \kappa \setminus \text{supp}(f)$  of cardinality  $|A| = |B| = n + 1$  and for any points  $x \in \text{supp}(f)$  and  $a \in A$ ,  $b \in B$  consider the even permutation  $\pi_{x,a,b} \in \text{Alt}_3(\kappa)$  with support  $\text{supp}(\pi_{x,a,b}) = \{x, a, b\}$  such that  $\pi_{x,a,b}(x) = a$ ,  $\pi_{x,a,b}(a) = b$  and  $\pi_{x,a,b}(b) = x$ . Since the topology  $\tau$  on  $\text{Alt}(\kappa)$  is shift-invariant and Hausdorff, the set  $O_{x,a,b} = \{g \in \text{Alt}(\kappa) : x_{x,a,b} \circ g \neq g \circ \pi_{x,a,b}\}$  is  $\tau$ -open. Taking into account that  $\pi_{x,a,b} \circ f(x) = f(x) \neq f(a) = f \circ \pi_{x,a,b}(x)$ , we conclude that the permutation  $f$  belongs to the  $\tau$ -open set  $O_{x,a,b}$  and  $O_f = \bigcap_{x \in \text{supp}(f)} \bigcap_{a \in A} \bigcap_{b \in B} O_{x,a,b}$  is a  $\tau$ -open neighborhood of  $f$ . We claim that the open set  $O_f$  has the desired property:  $O_f \cap \text{Alt}_n(\kappa) \subset \{g \in \text{Alt}(\kappa) : \text{supp}(g) = \text{supp}(f)\}$ . Take any permutation  $g \in O_f \cap \text{Alt}_n(\kappa)$ . Assuming that  $\text{supp}(g) \neq \text{supp}(f)$  and taking into account that  $|\text{supp}(g)| = |\text{supp}(f)| = n$ , we conclude that  $\text{supp}(f) \setminus \text{supp}(g)$  contains some point  $x$ . Since  $|A| = |B| > |\text{supp}(g)|$ , we can choose points  $a \in A \setminus \text{supp}(g)$  and  $b \in B \setminus \text{supp}(g)$ . Then  $\text{supp}(\pi_{x,a,b}) \cap \text{supp}(g) = \{x, a, b\} \cap \text{supp}(g) = \emptyset$  and hence  $\pi_{x,a,b} \circ g = g \circ \pi_{x,a,b}$ , which contradicts

the inclusion  $g \in O_f$ . This contradiction shows that the subspace  $O_f \cap \text{Alt}_n(\kappa) \subset \{g \in \text{Alt}(\kappa) : \text{supp}(g) = \text{supp}(f)\}$  is finite and hence discrete. Then the point  $f$  is isolated in the discrete  $\tau$ -open subset  $O_f \cap \text{Alt}_n(\kappa)$  of  $\text{Alt}_n(\kappa)$  and hence is isolated in  $\text{Alt}_n(\kappa)$ , witnessing that the subspace  $\text{Alt}_n(\kappa)$  of the topological space  $(\text{Alt}(\kappa), \tau)$  is discrete.  $\square$

In the following lemma we prove another equality of Theorem 2.

**Lemma 13.** *For any infinite cardinal  $\kappa$  and any group  $G$  with  $\text{Alt}(\kappa) \subset G \subset \text{Sym}(\kappa)$  we get  $\kappa = s_s(G)$ .*

*Proof.* Lemma 11 and obvious inequalities between the cardinal group invariants imply  $s_s(G) \leq w_l(G) = \kappa$ . It remains to prove that  $s_s(G) \geq \kappa$ . Assuming that  $s_s(G) < \kappa$  we could find a Hausdorff shift-invariant topology  $\tau$  with spread  $s(G, \tau) < \kappa$ . By Proposition 12, the subgroup  $\text{Alt}(\kappa)$  of  $G$  is  $\sigma$ -discrete in the topology  $\tau$ . Consequently,  $\text{Alt}(\kappa) = \bigcup_{i \in \omega} D_i$  where each subspace  $D_i$  is discrete in  $(G, \tau)$ . Since  $|\text{Alt}(\kappa)| = \kappa > s(G, \tau)$ , some set  $D_i$  has cardinality  $|D_i| > s(G, \tau)$ , which contradicts the definition of the spread  $s(G, \tau)$ . This contradiction shows that  $s_s(G) \geq \kappa$ .  $\square$

Establishing the equality  $\kappa = u_c(G)$  in Theorem 2 is the most difficult part of the proof, which requires some preparatory work.

For a cardinal  $\kappa$  and a subgroup  $G \subset \text{Sym}(\kappa)$  by  $\tau_p$  we denote the topology of point-wise convergence on  $G$ . This topology is generated by the subbase consisting of the sets

$$G(a, b) = \{g \in G : g(a) = b\} \text{ where } a, b \in \kappa.$$

By Theorem 2.1 of [3], on any group  $G$  with  $\text{Sym}_{\text{fin}}(\kappa) \subset G \subset \text{Sym}(\kappa)$ , the topology  $\tau_p$  coincides with the *restricted Zariski topology*  $\mathfrak{Z}'_G$  on  $G$ , generated by the subbase consisting of the sets

$$G \setminus \{a\}, \{x \in G : xbx^{-1} \neq aba^{-1}\} \text{ and } \{x \in G : (xcx^{-1})b(xcx^{-1})^{-1} \neq b\}$$

where  $a, b, c \in G$  and  $b^2 = c^2 = 1_G$ . It is easy to check that the topology  $\mathfrak{Z}'_G$  is shift-invariant. The following theorem generalizes Theorem 2.1 of [3].

**Theorem 14.** *Let  $\kappa$  be a cardinal. For any group  $G$  with  $\text{Alt}(\kappa) \subset G \subset \text{Sym}(\kappa)$  the topology  $\tau_p$  of pointwise convergence on  $G$  coincides with the restricted Zariski topology  $\mathfrak{Z}'_G$ .*

*Proof.* The proof of this theorem is just a suitable modification of the proof of Theorem 2.1 [3]. Fix a cardinal  $\kappa$  and a subgroup  $G \subset \text{Sym}(\kappa)$  containing the alternating group  $\text{Alt}(\kappa)$ . If the cardinal  $\kappa$  is finite, then so is the group  $G$ . In this case the  $T_1$ -topologies  $\tau_p$  and  $\mathfrak{Z}'_G$  coincide with the discrete topology on  $G$ . So, we assume that the cardinal  $\kappa$  is infinite.

For two distinct elements  $x, y \in \kappa$  consider the unique transposition  $t_{x,y}$  with support  $\text{supp}(t_{x,y}) = \{x, y\}$ . The transposition  $t_{x,y}$  exchanges  $x$  and  $y$  by their places and does not move other points of  $\kappa$ . For a subset  $A \subset \kappa$  consider the subgroups  $G(A) = \{g \in G : \text{supp}(g) \subset A\}$  and  $G_A = \{g \in G : \text{supp}(g) \cap A = \emptyset\} = \{g \in G : g|_A = \text{id}|_A\}$ .

**Lemma 15.** *For any 6-element subset  $A \subset \kappa$  the subgroup  $G_A$  is  $\mathfrak{Z}'_G$ -closed in  $G$ .*

*Proof.* Given any permutation  $f \in G \setminus G_A$  find a point  $a \in A$  with  $f(a) \neq a$ . Next, choose a point  $b \in A \setminus \{a, f(a)\}$  and two distinct points  $c, d \in A \setminus \{a, b, f(a), f(b)\}$ . Consider the even permutation  $t = t_{a,b} \circ t_{c,d} \in \text{Alt}(\kappa) \subset G$  and observe that  $t \circ f(a) = f(a) \neq f(b) = f \circ t(a)$ . Since  $\text{supp}(t) = \{a, b, c, d\} \subset A$ , the transposition  $t$  commutes with all permutations  $g \in G_A$ , which implies that

$$O_f = \{g \in G : t \circ g \neq g \circ t\} = \{g \in G : tgt^{-1} \neq g\}$$

is a  $\mathfrak{Z}'_G$ -open neighborhood of  $f$ , disjoint with the subgroup  $G_A$ .  $\square$

**Lemma 16.** *For each 6-element subset  $A \subset \kappa$  the subgroup  $G_A$  is  $\mathfrak{Z}'_G$ -open in  $G$ .*

*Proof.* To derive a contradiction, assume that for some 6-element set  $A' \subset \kappa$  the subgroup  $G_{A'}$  is not  $\mathfrak{Z}'_G$ -open. Being  $\mathfrak{Z}'_G$ -closed, this subgroup is nowhere dense in the semi-topological group  $(G, \mathfrak{Z}'_G)$ . Observe that for any 6-element subset  $A \subset \kappa$  and any even permutation  $f \in \text{Alt}(G)$  with  $f(A) = A'$ , we get  $G_A = f^{-1} \circ G_{A'} \circ f$ , which implies that the subgroup  $G_A$  is closed and nowhere dense in  $(G, \mathfrak{Z}'_G)$ .

We claim that for every 6-element set  $A \subset \kappa$  and any finite subset  $B \subset \kappa$  the set  $G(A, B) = \{g \in G : g(A) \subset B\}$  is nowhere dense in  $(G, \mathfrak{Z}'_G)$ . Since  $A$  and  $B$  are finite, we can choose a finite set  $F \subset G$  such that for any  $g \in G(A, B)$  there exists  $f \in F$  with  $g|_A = f|_A$ . Then  $f^{-1} \circ g|_A = \text{id}_A$  which implies that  $G(A, B) = \bigcup_{f \in F} f \circ G_A$  is nowhere dense in  $(G, \mathfrak{Z}'_G)$ .

Now fix any four pairwise disjoint 6-element subsets  $A_1, A_2, A_3, A_4 \subset \kappa$  and consider their union  $A = \bigcup_{i=1}^4 A_i$ . Consider the finite subset  $T = \{t_{a_1, a_2} \circ t_{a_3, a_4} : (a_i)_{i=1}^4 \in \prod_{i=1}^4 A_i\} \subset \text{Alt}(\kappa) \subset G$ . For any permutations  $t, s \in T$  with  $t \circ s \neq s \circ t$  the set

$$U_{t,s} = \{u \in G : (usu^{-1})t(usu^{-1})^{-1} \neq t\}$$

is a  $\mathfrak{Z}'_G$ -open neighborhood of  $1_G$  by the definition of the topology  $\mathfrak{Z}'_G$ . Since  $T$  is finite, the intersection

$$U = \bigcap \{U_{t,s} : t, s \in T, ts \neq st\}$$

is a  $\mathfrak{Z}'_G$ -open neighborhood of  $1_G$ . Choose a permutation  $u \in U$  which does not belong to the nowhere dense subset  $\bigcup_{i=1}^4 G(A_i, A)$ . For every  $i \in \{1, 2, 3, 4\}$  there is a point  $a_i \in A_i$  such that  $u(a_i) \notin A$ . Choose any point  $a'_2 \in A_2 \setminus \{a_2\}$  and consider the non-commuting permutations  $t = t_{a_1, a'_2} \circ t_{a_3, a_4}$  and  $s = t_{a_1, a_2} \circ t_{a_3, a_4}$  in  $T$ . It follows from  $u \in U$  that the permutation  $v = usu^{-1}$  does not commute with the permutation  $t$ . On the other hand, the support  $\text{supp}(v) = \text{supp}(usu^{-1}) = u(\{a_1, a_2, a_3, a_4\})$  does not intersect the set  $A \supset \{a_1, a'_2, a_3, a_4\} = \text{supp}(t)$ , which implies that  $vt = tv$ . This contradiction completes the proof of Lemma 16.  $\square$

Now we are able to prove that  $\mathfrak{Z}'_G = \tau_p$ . Taking into account that  $\tau_p$  is a Hausdorff group topology on  $G$ , we conclude that  $\mathfrak{Z}'_G \subset \tau_p$ . To prove the reverse inclusion, it suffices to check that for every  $a, b \in \kappa$  the subbasic  $\tau_p$ -open set  $G(a, b) = \{g \in G : g(a) = b\}$  is  $\mathfrak{Z}'_G$ -open. Choose any 6-element subset  $A \subset \kappa$  containing the point  $a$ . By Lemma 16 the subgroup  $G_A$  is  $\mathfrak{Z}'_G$ -open and hence for any  $g \in G(a, b)$  the set  $g \circ G_A \subset G(a, b)$  is a  $\mathfrak{Z}'_G$ -open neighborhood of  $g$ , witnessing that the set  $G(a, b)$  is  $\mathfrak{Z}'_G$ -open.  $\square$

Combining Theorem 14 with Proposition 4.1 of [3], we obtain the following generalization of Theorem 4.2 of [3].

**Theorem 17.** *Let  $\kappa$  be an infinite cardinal and  $G$  be a group such that  $\text{Alt}(\kappa) \subset G \subset \text{Sym}(\kappa)$ . Each shift-invariant  $T_1$ -topology  $\tau$  with separately continuous commutator on  $G$  contains the topology  $\tau_p$  of pointwise convergence on  $G$ .*

Now we are able to prove the final equality of Theorem 2.

**Lemma 18.** *For any infinite cardinal  $\kappa$  and any group  $G$  with  $\text{Alt}(G) \subset G \subset \text{Sym}(G)$  we get  $\kappa = u_c(G)$ .*

*Proof.* It follows from Lemma 11 that  $u_c(G) \leq w_l(G) = \kappa$ . So, it remains to prove that  $u_c(G) \geq \kappa$ . To derive a contradiction, assume that  $u_c(G) < \kappa$  and find a topology  $\tau \in \mathcal{T}_c(G)$  such that  $u(G, \tau) < \kappa$ . By Theorem 17,  $\tau_p \subset \tau$ , which implies that the subgroup  $G_0 = \{g \in G : g(0) = 0\}$  is  $\tau$ -open. For every  $i \in \kappa \setminus \{0\}$  fix a permutation  $g_i \in \text{Alt}(\kappa)$  such that  $g_i(0) = i$  and observe that for every  $0 < i \neq j < \kappa$  we get  $g_j \notin g_i \circ G_0$ , which means that the set  $D = \{g_i\}_{0 < i < \kappa}$  is uniformly discrete in  $(G, \tau)$ . So,  $u(G, \tau) \geq \kappa$ , which is a desired contradiction.  $\square$

**Problem 19.** *Let  $\kappa$  be a cardinal and  $G$  be a group with  $\text{Alt}(\kappa) \subset G \subset \text{Sym}(\kappa)$ . Calculate the value of the cardinal invariant  $u_s(G)$ . Is  $u_s(G) = \kappa$ ?*

The equalities in Theorem 2 hold also for some other groups, in particular for the additive group  $\mathbb{R}$  of real numbers.

**Example 20.** *The group  $\mathbb{R}$  does admit a linear group topology with countable weight and hence has  $u_s(\mathbb{R}) = w_l(\mathbb{R}) = \text{cn}(\mathbb{R}) = \omega$ .*

The equalities in this example follows from our next theorem evaluating the cardinal group invariants of infinite abelian groups.

**Theorem 21.** *Each infinite abelian group  $G$  has*

$$\log \log |G| \leq s_s(G) \leq w_s(G) = w_l(G) = \log |G| \quad \text{and} \quad u_s(G) = u_l(G) = cn(G) = \omega.$$

*Proof.* The inequalities  $s_s(G) \leq w_s(G) \leq w_l(G)$  hold for any infinite group  $G$ . The lower bounds  $\log \log |G| \leq s(G)$  and  $\log |G| \leq w(G)$  follow from the inequalities  $|X| \leq 2^{2^{s(X, \tau)}}$  and  $|X| \leq 2^{w(X, \tau)}$  holding for any Hausdorff topological space  $(X, \tau)$  (see Theorems 5.5 and 3.1 in [4]).

Now we prove that  $w_l(G) \leq \log |G|$ . Since each infinite abelian group embeds into a divisible abelian group of the same cardinality [8, 4.1.6], we can assume that  $G$  is divisible (which means that for every  $x \in G$  and  $n \in \mathbb{N}$  there exists  $y \in G$  such that  $y^n = x$ ). It is clear that the additive group  $\mathbb{Q}_0$  of rational numbers is divisible and so is the quasi-cyclic  $p$ -group  $\mathbb{Q}_p = \{z \in \mathbb{C} : \exists k \in \mathbb{N} \ z^{p^k} = 1\}$  for any prime number  $p$ . Let  $\mathbb{P}$  denote the set of all prime numbers and  $\mathbb{P}_0 = \{0\} \cup \mathbb{P}$ . Endow the countable groups  $\mathbb{Q}_p$ ,  $p \in \mathbb{P}_0$ , with the discrete topologies and for the cardinal  $\kappa = \log |G|$  consider the Tychonoff product  $L = \prod_{p \in \mathbb{P}_0} \mathbb{Q}_p^\kappa$ . Taking into account that  $L$  is a linear topological group of weight  $\kappa$ , we conclude that  $w_l(L) \leq \kappa$ . It remains to prove that the divisible group  $G$  embeds into  $L$ .

For a cardinal  $\lambda$  and an abelian group  $A$  by  $A^{(\lambda)}$  we denote the direct sum of  $\lambda$  many copies of the groups  $A$ . By [8, 4.1.5], the group  $G$ , being divisible, is isomorphic ( $\approx$ ) to the direct sum  $\bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(k_p)}$  for some cardinals  $k_p \leq |G|$ ,  $p \in \mathbb{P}_0$ . By the same reason, for every  $p \in \mathbb{P}_0$  the  $\kappa$ -th power  $\mathbb{Q}_p^\kappa$  of  $\mathbb{Q}_p$  is isomorphic to the direct sum  $(\mathbb{Q}_0 \oplus \mathbb{Q}_p)^{(2^\kappa)}$  of  $2^\kappa$  many copies of the groups  $\mathbb{Q}_0 \oplus \mathbb{Q}_p$ . Since  $k_p \leq |G| \leq 2^\kappa$  for all  $p \in \mathbb{P}_0$ , the group  $G \approx \bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(k_p)}$  embeds into the direct sum  $\bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(2^\kappa)} \approx L$ . Now the monotonicity of the cardinal group invariant  $w_l$  ensures that  $w_l(G) \leq w_l(L) \leq \kappa = \log |G|$ . Combining this inequality with  $w_s(G) \geq \log |G|$ , we get the equalities  $w_s(G) = w_l(G) = \log |G|$ .

Next, we prove that  $u_s(G) = u_l(G) = \omega$ . Since  $\omega \leq u_s(G) \leq u_l(G)$ , it suffices to show that  $u_l(G) \leq \omega$ . Since the cardinal group invariant  $u_l$  is monotone, we can assume that the group  $G$  is divisible and hence is isomorphic to the direct sum of countable abelian groups. This implies that the trivial subgroup of  $G$  can be written as the intersection of subgroups of at most countable index in  $G$ . By Proposition 5,  $u_l(G) \leq \omega$ .

Finally, we prove that  $cn(G) = \omega$ . It suffices to prove that  $G$  satisfies the weak  $\omega^+$ -compatibility condition. Fix a finite group  $F$  and isomorphisms  $f_i : F \rightarrow F_i$ ,  $i \in \omega_1$ , onto subgroups  $F_i \subset G$ . The  $\Delta$ -Lemma [5, 9.18] yields an uncountable subset  $\Omega \subset \omega_1$  and a finite subgroup  $D \subset G$  such that  $F_i \cap F_j = D$  for any distinct indices  $i, j \in \Omega$ . By the Pigeonhole Principle, for some subgroup  $E \subset F$  the set  $\Omega_1 = \{i \in \Omega : f_i^{-1}(D) = E\}$  is uncountable. Since the set of isomorphisms from  $E$  to  $D$  is finite, for some isomorphism  $f : E \rightarrow D$  the set  $\Omega_2 = \{i \in \Omega_1 : f_i|_E = f\}$  is uncountable. Now take any two ordinals  $i < j$  in the set  $\Omega_2$  and consider the subgroup  $F_{ij} = F_i + F_j$  of  $G$ . Define a homomorphism  $\phi : F_{ij} \rightarrow F$  assigning to a point  $x = x_i + x_j$  with  $x_i \in F_i$  and  $x_j \in F_j$  the point  $\phi(x) = f_i^{-1}(x_i) + f_j^{-1}(x_j)$ . Let us show that the map  $\phi$  is well-defined. Assume that  $x = x'_i + x'_j$  for some points  $x'_i \in F_i$  and  $x'_j \in F_j$ . Then  $0 = x - x = (x_i - x'_i) + (x_j - x'_j)$  implies that  $x_i - x'_i = x'_j - x_j \in F_i \cap F_j = D$  and hence

$$f_i^{-1}(x_i) - f_i^{-1}(x'_i) = f_i^{-1}(x_i - x'_i) = f^{-1}(x_i - x'_i) = f^{-1}(x'_j - x_j) = f_j^{-1}(x'_j - x_j) = f_j^{-1}(x'_j) - f_j^{-1}(x_j)$$

and finally  $f_i^{-1}(x_i) + f_j^{-1}(x_j) = f_i^{-1}(x'_i) + f_j^{-1}(x'_j)$ . Therefore the map  $\phi : F_{ij} \rightarrow F$  is a well-defined homomorphism such that  $\phi \circ f_i = \phi \circ f_j = \text{id}_F$ , witnessing that the group  $G$  satisfies the weak  $\omega^+$ -compatibility condition and  $wc(G) = \omega$ .  $\square$



Theorem 21 implies that for infinite abelian groups the diagram describing the relations between the cardinal group invariants takes the following form:

$$\begin{array}{ccccccc}
 & & w_s & \equiv & w_c & \equiv & w_g & \equiv & w_l & \equiv & \log |\cdot| \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 \log \log |\cdot| & \longrightarrow & s_s & \longrightarrow & s_c & \longrightarrow & s_g & \longrightarrow & s_l & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & u_s & \equiv & u_c & \equiv & u_g & \equiv & u_l & \equiv & cn = \omega
 \end{array}$$

Looking at this diagram it is natural to ask about the values of the cardinal invariants in the middle row. Are they equal to  $\log \log |G|$  or to  $\log |G|$ ? Surprisingly, the answer depends on additional set-theoretic assumptions.

**Theorem 22.** (1) *PFA implies that each group  $G$  of cardinality  $|G| > \mathfrak{c}$  has  $s_s(G) > \omega = \log \log \mathfrak{c}^+$ .*  
 (2) *For any regular uncountable cardinals  $\lambda \leq \kappa$  it is consistent that  $2^\omega = \lambda$ ,  $2^{\omega_1} = \kappa$  and each infinite abelian group  $G$  of cardinality  $|G| \leq \kappa$  has linear spread  $s_l(G) = \omega = \log \log |G|$ .*

*Proof.* 1. The first statement follows from a result of Todorcevic [10, 8.12] saying that under PFA each Hausdorff space  $X$  of countable spread has cardinality  $|X| \leq \mathfrak{c}$ .

2. By [6, 4.10] for any regular uncountable cardinals  $\lambda \leq \kappa$  it is consistent that  $2^\omega = \lambda$ ,  $2^{\omega_1} = \kappa$  and there exists a subspace  $X \subset \{0,1\}^\lambda$  of cardinality  $|X| = \kappa$  such that each finite power  $X^n$  is hereditarily separable and hence has countable spread. The following lemma completes the proof of the statement (2).  $\square$

**Lemma 23.** *Let  $X$  be a zero-dimensional space such that each finite power  $X^n$  of  $X$  has countable spread. Then any infinite abelian group  $G$  of cardinality  $|G| \leq |X|$  has linear spread  $s_l(G) = \omega$ .*

*Proof.* Consider the space  $C_p(X, \mathbb{Q}) \subset \mathbb{Q}^X$  of continuous functions from  $X$  to the discrete group  $\mathbb{Q}$  of rational numbers, and let  $C_p C_p(X, \mathbb{Q})$  be the space of continuous functions from  $C_p(X, \mathbb{Q})$  to  $\mathbb{Q}$ . Observe that the topology on  $C_p C_p(X, \mathbb{Q})$  inherited from the Tychonoff product  $\mathbb{Q}^{C_p(X, \mathbb{Q})}$  is linear.

The zero-dimensionality of  $X$  can be used to prove that the map  $\delta : X \rightarrow C_p C_p(X, \mathbb{Q})$  assigning to each  $x \in X$  the Dirac measure  $\delta_x : C_p(X, \mathbb{Q}) \rightarrow \mathbb{Q}$ ,  $\delta_x : f \mapsto f(x)$ , is a topological embedding. Let  $L$  be the  $\mathbb{Q}$ -linear hull of the set  $\delta(X)$  in the  $\mathbb{Q}$ -linear space  $C_p C_p(X, \mathbb{Q})$  and  $Z$  be the group hull of  $\delta(X)$  in  $C_p C_p(X, \mathbb{Q})$ . It can be shown that  $Z$  is a closed subgroup of  $L$ , so we can consider the quotient group  $L/Z$  and notice that it is a divisible torsion group isomorphic to the direct sum  $\bigoplus_{p \in \mathbb{P}} \mathbb{Q}_p^{(|X|)}$ . Here  $\mathbb{P}$  is the set of prime numbers and  $\mathbb{Q}_p = \{z \in \mathbb{C} : \exists k \in \mathbb{N} z^{p^k} = 1\}$  for  $p \in \mathbb{P}$ . Consequently, the group  $L \times (L/Z)$  is isomorphic to  $\bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(|X|)}$  where  $\mathbb{P}_0 = \mathbb{P} \cup \{0\}$  and  $\mathbb{Q}_0 = \mathbb{Q}$ . It can be shown that the space  $L \times (L/Z)$  can be written as the countable union of spaces homeomorphic to continuous images of finite powers of  $X$ , which implies that the space  $L \times (L/Z)$  has countable spread.

Since the group  $L \times (L/Z)$  carries a linear Hausdorff group topology of countable spread, so does the group  $\bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(|X|)}$ . By [8, 4.1.5 and 4.1.6], every infinite abelian group  $G$  of cardinality  $|G| \leq |X|$  embeds into  $\bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(|G|)} \subset \bigoplus_{p \in \mathbb{P}_0} \mathbb{Q}_p^{(|X|)}$  and hence  $G$  carries a linear Hausdorff group topology with countable spread, which implies that  $s_l(G) = \omega$ .  $\square$

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